Strategic Party Heterogeneity

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Abstract
Most spatial voting models assume or imply that homogeneous candidate teams offer parties the best prospect for winning elections. Yet, we know that candidates from the same political party often adopt different policy positions. This article reconciles theory and reality by identifying a strategic rationale for political parties to recruit a diverse pool of candidates. I develop a spatial model in which two parties each select a distribution of candidates to compete in an upcoming election. The model demonstrates that parties positioned close to the median voter should field a more homogeneous set of candidates than parties with platforms that are more distant. I test this prediction with data on the policy positions of Democratic and Republican candidates for congressional and state legislative elections since 1990. In line with the model’s predictions, I find that minority parties – presumably more distant from the median voter – are more heterogeneous than majority parties.

Introduction
Political parties are inherently heterogeneous. Many of their candidates campaign on – and if elected, act on – positions that vary significantly from one another and from the party platform. Consider, for example, the broad array of viewpoints held by the Republican Party’s presidential hopefuls during the 2012 primaries. On the issue of immigration alone, the candidates’ positions range from supporting legal residence for undocumented workers with long-standing ties to the U.S. (Gingrich), to ending birthright citizenship for children whose parents crossed the border illegally.
The degree of heterogeneity varies both within and between the two parties over time. In the second half of the 20th century, the Democrats became more homogeneous as conservative southerners left the party to support the Republicans. And after losing the 2008 election, the emerging Tea Party movement made the Republican Party more heterogeneous (and more competitive). A party’s heterogeneity has important consequences for its ability to appeal to a diverse set of constituents, offer voters a coherent message, win elections, and ultimately enact its program in office (Ashworth and Bueno de Mesquita 2008, Downs 1957, Grynaviski 2010, Rohde 1991, Snyder and Ting 2002, Woon and Pope 2008).

Considering its impact on electoral and legislative politics, it is surprising that models of electoral competition largely overlook heterogeneity. The Downsian spatial voting model depicts parties as unconstrained unitary actors who can freely change policy positions to match voters’ preferences (Downs 1957). Extensions to the model follow the same logic, seeing parties and their candidates as homogeneous teams occupying a single policy stance. The few exceptions view heterogeneity as an electoral liability. Parties only cast a wide net if imposing conformity is too costly or beyond the party’s control (Ashworth and Bueno de Mesquita 2008, Snyder and Ting 2002).

This article offers an electoral rationale for party heterogeneity. By recruiting a diverse set of candidates, parties may increase their appeal to voters whose positions differ from the party platform. Consequently, parties with positions that are out of step with the median voter should field a heterogeneous candidate pool in order to maximize their chances of winning the election.

In the simple model that follows, two parties each choose a pool of potential candidates to compete in an upcoming election. Each party then nominates a candidate from their respective pool, and voters select the candidate who is closest to their ideal point on a single policy dimension. In contrast with the Downsian model, parties cannot always align their platform (and candidates)

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1Of course, heterogeneity among legislators largely reflects variation in voters’ preferences across districts. Yet, geographic variation cannot explain differences in heterogeneity between the two parties.
with the position of the median voter. Rather, parties control their set of potential candidates by specifying distributional parameters – a mean and variance – describing these candidates’ positions. A party can “screen” out potential extremists by selecting a more homogeneous pool, or it can act as a broad coalition by allowing candidates with varying viewpoints to seek the nomination.

I find that when parties can freely change their platform – that is, their mean position – and heterogeneity, they should move to the position of the median voter for the same reasons laid out by Hotelling (1929), Black (1948) and Downs (1957), and adopt perfect homogeneity. However, when parties are constrained from converging to the median voter (as they most often are), they should field a more heterogeneous set of candidates. Specifically, I find that there exists a unique level of heterogeneity that maximizes a party’s chance of winning the election, given its platform location and the distribution of candidates from the opposing party. In equilibrium, the party that is closer to the median voter should adopt a narrower distribution than its more distant opponent.

I test this prediction for congressional and state legislative elections from 1990 to 2010 and find support for the theory: majority parties – presumably closer to the median voter – are significantly more likely to be more homogeneous than minority parties.

This article presents a different view of party heterogeneity than that found in current literature. Previous research focuses on the link between heterogeneity and voter uncertainty, arguing that risk-averse voters are less likely to support heterogeneous parties because their labels, or “brand names,” are ambiguous or variable (Ashworth and Bueno de Mesquita 2008, Grynaviski 2010, Snyder and Ting 2002, Woon and Pope 2008). While this research justifies cases of party homogeneity, it cannot explain strategic party heterogeneity, or why the level of heterogeneity differs over time or between the two major parties. And yet the anecdotal record frequently finds politicians “expanding the

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2 Parties rarely converge to the median voter (Ansolabehere, Snyder and Stewart 2001, Burden 2004a, Frendreis et al. 2003, Jessee 2010). Divergence may reflect a variety of factors absent in the Downsian model, including policy-minded candidates, valuable activists with more extreme positions, or primary elections (Aldrich 2011, Grofman 2004).
base” or constructing “a big tent.” A related area of research argues that parties or candidates may deliberately obfuscate their positions. This strategy makes sense when voters are risk seeking or hold “intense” preferences (Aragonès and Postlewaite 2002, Shepsle 1972), when ambiguity can alter the salience that voters place on issues (Page 1976), or when politicians plan to enact their preferred policy position once in office (Alesina and Cukierman 1990, Aragonès and Neeman 2000). In contrast, the model presented in this article finds that purely office-seeking parties may be strategically heterogeneous even when voters are averse to risk.

The next section develops the formal heterogeneity model. I then derive the results of the paper: optimal platform location (given heterogeneity), optimal heterogeneity (given platform location), and Nash equilibria. After presenting the comparative statics and equilibria, I find that the model’s predictions are robust to multiple extensions, including adding informative party labels and voter uncertainty. The model’s primary implication – that parties closer to the median voter should be more homogeneous than their more distant competitors – is then tested using a unique data set developed by Bonica (2010), which identifies the positions of incumbents and challengers in national and state legislative elections. I conclude by exploring the model’s implications for legislative behavior, as well as discussing future avenues for research.

A Spatial Model of Party Heterogeneity

Two parties, $i$ and $j$, each seek to maximize their probability of winning an upcoming election. Both parties choose a platform along a single policy dimension, where the platform of party $p \in \{i, j\}$ is denoted $\mu_p$. They also choose a pool of potential candidates who may compete in the election. The positions of the candidates in party $p$’s pool follow a distribution, $f_p$, such that the mean of the candidate distribution equals the party platform: $E[f_p] = \mu_p$. To determine the variation of candidates’ positions around the platform, parties set a level of heterogeneity, $\sigma_p$, which equals the
standard deviation of the candidate distribution: \( \sigma_p = \sqrt{\mathbb{E}[(f_p - \mu_p)^2]} \). Parties wishing to exclude candidates who are far from the platform will impose greater homogeneity, or “screening,” and set \( \sigma_p \) close to zero. Parties hoping to attract candidates with diverse positions will create a “big tent,” and choose a high level of heterogeneity.

After selecting their respective candidate pools, each party nominates a candidate to run in the upcoming election. The nomination process is modeled as a random draw from the party’s candidate pool.\(^3\) Both candidates truthfully announce the policy positions they will pursue if they are elected, and voters observe these positions with certainty. (Later I relax both these assumptions.) Voters’ ideal points fall along the same policy dimension as candidates’ positions. A voter’s utility from the policy pursued by the candidate in party \( p \) is given by \( u(|v - c_p|) \), where \( v \) is the voter’s ideal point, \( c_p \) is the candidate’s policy position in party \( p \), and \( u \) is any function that decreases in \( [0, \infty) \), such as a quadratic or linear loss function.\(^4\) Thus, as in the classic spatial model, voters choose the candidate closest to their ideal point. The candidate receiving a majority of votes wins the election.

The most important distinction between this and the Downsian model is that parties can only exert indirect control over the positions of their nominees by setting the distributional parameters – their platform (or mean) and level of heterogeneity (or standard deviation) – that define the set of eligible candidates.

**Election Outcomes**

In two-party competition with full information and sincere voting, the candidate who captures the vote of the median voter wins the election. Hence, the discussion can be simplified by stipulating

\(^3\)Ashworth and Bueno de Mesquita (2008) and Snyder and Ting (2002) also assume that candidates are nominated by random draw from a distribution of potential candidates.

\(^4\)The necessary assumption is that voters select candidates based on their relative proximity along a single dimension, but this dimension need not represent policy. It could instead reflect partisanship, ideology, or a single issue. Later, I examine how the model changes when voters care about party platforms as well as candidate positions.
that the parties compete for the support of a single voter who represents the median in the electorate, and by setting this voter’s ideal point to zero. Given the candidate pools, \( f_i \) and \( f_j \), the probability that the median voter chooses the candidate from party \( i \), \( P_i \), is

\[
P_i = \int_{-\infty}^{\infty} f_j(c_j) \int_{-|c_j|}^{|c_j|} f_i(c_i) \, dc_i \, dc_j. \tag{1}
\]

For a given candidate from party \( j \) with position \( c_j \), \( P_i \) is simply the probability that \( c_i \) lies between \(-|c_j|\) and \(|c_j|\), as illustrated in Figure 1. The probability that \( c_i \) is closer to the voter than any \( c_j \) is found by simply integrating \( \int_{-|c_j|}^{|c_j|} f_i(c_i) \, dc_i \) over all possible values of \( c_j \).

To solve for the parties’ optimal strategies, I assume that each party’s candidate pool, \( f_p \), is symmetric and single-peaked. I also make the additional technical assumption that \( \frac{f_p(a/\sigma)}{f_p(b/\sigma)} \) is monotonic in \( \sigma > 0 \) for all \( a \neq b \).\(^5\) These assumptions are fairly unrestrictive: candidate preferences may follow most common symmetric distributions, such as the normal or logistic distribution. I also assume that \( f_p \) can be fully described by its mean and variance, allowing each party’s distribution to be written in terms of a single standard distribution, \( \phi \), with mean zero and standard deviation one: \( f_p(c_p) = \frac{1}{\sigma_p} \phi\left( \frac{c_p - \mu_p}{\sigma_p} \right) \). I assume that when \( \sigma_p = 0 \), \( f_p \) equals a delta distribution at \( \mu_p \): \( \delta(\mu_p) = \lim_{\sigma_p \to 0} \frac{1}{\sigma_p} \phi\left( \frac{c_p - \mu_p}{\sigma_p} \right) \). Substituting this notation into equation (1) gives:

\[
P_i = \int_{-\infty}^{\infty} \frac{1}{\sigma_j} \phi\left( \frac{c_j - \mu_j}{\sigma_j} \right) \int_{-|c_j|}^{|c_j|} \frac{1}{\sigma_i} \phi\left( \frac{c_i - \mu_i}{\sigma_i} \right) \, dc_i \, dc_j. \tag{2}
\]

The next section examines how changes in the parameters from equation (2) shape election outcomes and party strategies.

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\(^5\)This assumption essentially implies that \( f_p \) changes concavity only one time on either side of its mean and guarantees that the optimal level of heterogeneity is unique.
Figure 1: Two parties and their distributions of candidates. The probability that party $i$ wins the vote of a voter positioned at zero is equal to the area under party $i$'s distribution between $-|c_j|$ and $|c_j|$. This area is shaded in the graph for a particular $c_j$. 
Comparative Statics and Equilibria

While previous research has focused on strategic changes in platform location when parties are perfectly homogeneous, the model presented here investigates strategic changes in heterogeneity when platforms are fixed. Before turning to this, however, it is useful to ask if heterogeneity affects a party’s optimal platform location. I show below that the simple answer is no. Heterogeneous parties should follow the Downsian prescription and converge to the median voter. But several interesting results do arise; in particular, a shift in platform now has a probabilistic effect on winning that depends not only on a party’s relative proximity to the median voter, but also on its heterogeneity.

Shifts in a Party’s Platform

This section relaxes the standard assumption that parties are homogeneous party teams and examines how platform location affects success.

Proposition 1. A party’s probability of winning increases as its platform moves closer to the median voter.

(Formal Statement: For all $i \neq j$, decreasing $|\mu_i|$ increases $P_i$ and decreases $P_j$.)

The proofs for Propositions 1 through 8 are available in the Appendix. In the standard spatial model, where parties are represented by points instead of distributions, a shift in a party’s position only affects election outcomes if it changes which party is closest to the median voter. Proposition 1 demonstrates that when candidate positions are probabilistic, any platform shift toward the median voter increases a party’s probability of winning the election. A party described by a symmetric, single-peaked distribution maximizes its probability of winning the election by setting its platform at the position of the median voter, regardless of its level of heterogeneity.\(^6\)

\(^6\)It is worth noting that this result does not hold for all distributions of party $i$. If party $i$’s distribution is
Proposition 1 demonstrates that a party’s chance of winning the election diminishes as its platform shifts away from the median voter. But the degree to which this shift hurts the party’s success depends on its initial platform location, as well as its heterogeneity.

Proposition 2. Shifting a party’s platform away from the median voter decreases its probability of winning:

a. less when its platform is very far from or very close to the median voter, and more when it is somewhere in between. (Formal Statement: The marginal effect, \(\frac{\partial P_i}{\partial |\mu_i|}\), approaches zero from below as \(|\mu_i|\) approaches infinity and as \(|\mu_i|\) approaches zero.)

b. less when it is very heterogeneous, and by twice the density of its opponent’s candidates at its platform when it is very homogeneous. (Formal Statement: The marginal effect, \(\frac{\partial P_i}{\partial |\mu_i|}\), approaches zero from below as \(\sigma_i\) approaches infinity and approaches \(-2f_j(|\mu_i|)\) as \(\sigma_i\) approaches zero.)

Changes in platform location matter least when parties are very close to the median voter (and therefore are likely to win the election) or very far from the median voter (and are highly unlikely to win). Parties located somewhere in between face the stiffest competition, and they therefore have the most to gain or lose by adjusting their platform. If platform movement is costly, we may expect those parties that are neither very far nor very close to the median voter to be most likely to adjust their platform locations.

Proposition 2b describes how a party’s heterogeneity influences the marginal effect of a change in party platform. If a party’s heterogeneity is sufficiently high, its probability of winning approaches zero, and a change in platform has little effect. If a party is perfectly homogeneous, its chance of winning is simply the probability that its platform is closer to the median voter than the candidate from the other party (i.e. \(1 - \int_{-|\mu_i|}^{|\mu_i|} f_j(c_j) \, dc_j\)). If the opposition party is also perfectly homogeneous, an a small change in platform location only matters if the two parties are equidistant from the median.

Asymmetric or has more than one mode, \(P_i\) is not always maximized when the party’s mean position matches that of the median voter. In contrast, Proposition 1 does not depend on the opposition party’s distribution of candidates. For any distribution, \(f_j\), party \(i\) maximizes its probability of winning by converging to the median voter.
Because the probability that party $j$ wins the election is $1 - P_i$, these results apply to the opposing party as well. Shifts in either party’s platform affect election outcomes more when the party changing platform locations is neither very close nor very far from the median voter. Changes in the party platform have the smallest effect when heterogeneity is high.

**Party Heterogeneity**

The previous section demonstrates that heterogeneity does not alter a party’s vote-maximizing platform position: parties should always converge to the median voter. But what happens when platforms are off median? In reality, party platforms and candidate positions rarely converge to the position of the median voter (Ansolabehere, Snyder and Stewart 2001, Burden 2004a, Frendreis et al. 2003, Jessee 2010). Parties adopt more extreme positions to woo party activists (Aldrich 1983, Aldrich and McGinnis 1989, Frendreis et al. 2003) and primary voters (Burden 2004a, Owen and Grofman 2006), or to deter abstention (Downs 1957) or the entry of an independent or third-party candidate (Lee 2011). Candidates may also have policy preferences of their own that are more extreme than those of the electorate (Aldrich 2011, Calvert 1985, Wittman 1983). And even those parties that successfully locate at the position of the median voter may find themselves unable to quickly adjust their platform when exogenous shocks (such as terrorist attacks or redistricting) dramatically change the partisan landscape and shift the position of the median voter (Burden 2004a). Thus, it is important to ask if increasing heterogeneity can offer parties an attractive alternative to adjusting their policy platform. Should parties that are disadvantaged by unpopular platforms choose to be strategically heterogeneous? And if so, what distribution will maximize their success?
Figure 2: The distribution of candidates for three possible values of $\sigma_i$ and a fixed $c_j$. Party $i$ maximizes the probability that it selects a candidate in $[-|c_j|, |c_j|]$ when its heterogeneity equals $\sigma_{i2}$. 
For intuition, Figure 2 depicts a scenario where party $i$’s platform ($\mu_i$) is farther from the median voter than its opposing party’s nominee. (In the example, $c_j$ is fixed.) The figure also shows three candidate distributions for party $i$. Its platform is identical in each; the only variation is in its heterogeneity, where $\sigma_{i1} < \sigma_{i2} < \sigma_{i3}$. As we can see, the density between $-|c_j|$ and $|c_j|$ is greatest for the distribution with a medium variance ($\sigma_{i2}$). If the party adopts a low variance (e.g. $\sigma_{i1}$), it can improve its chances by widening the candidate pool. But a pool that is too diverse (e.g. $\sigma_{i3}$) weakens its chance of winning.

[Figure 2 about here]

This logic underpins the main result in the paper. A homogeneous party positioned far from the median voter has no chance of winning. By diversifying its candidate pool, the party increases its likelihood of nominating a candidate closer to the voter than the opposing party’s nominee. Yet, too much heterogeneity quickly becomes a liability. Beyond a certain point, increasing heterogeneity decreases the probability that a party’s candidate is closer to the voter than its opponent.

**Proposition 3.** There is a unique level of heterogeneity that maximizes a party’s probability of winning, given its platform location and the distribution of candidates for the opposing party.

(Formal Statement: For a fixed $\mu_i$ and $f_j$, where $f_j \neq \delta(0)$, there exists a unique value of $\sigma_i$, $\sigma_i^*$, that maximizes $P_i$.)

The proof demonstrates that there is one and only one level of heterogeneity that maximizes a party’s chance of winning, given its platform and competitor’s distribution. A party’s optimal level of heterogeneity will depend on its distance from the median voter as well as the distribution of candidates in its competitor’s party.

7When $f_j = \delta(0)$ and $\mu_i = 0$, $\sigma_i^* = 0$. When $f_j = \delta(0)$ and $\mu_i \neq 0$, any value of $\sigma_i$ is optimal.
To see this graphically, Figure 3 plots the probability that party $i$ wins the election as a function of its standard deviation, $\sigma_i$, for varying values of its platform location, $\mu_i$, and a fixed distribution for party $j$. The top line in the graph depicts party $i$’s probability of winning when its platform coincides with the median voter’s position. Each consecutive line below this depicts its probability of winning as its platform becomes increasingly distant. For each platform, the probability of winning, $P_i$, is a single-peaked function of heterogeneity ($\sigma_i$).

[Figure 3 about here]

As we can see, perfect homogeneity is not always the best strategy. The optimal level of heterogeneity depends on a party’s relative distance to the voter, but for parties with highly unpopular platforms (e.g. $\mu_i = 8$), heterogeneity is a party’s only chance for electoral success. This result differs significantly from previous research, which assumes that party heterogeneity is always electorally costly and that parties will institute strict screening methods if they can afford to do so (Ashworth and Bueno de Mesquita 2008).\(^8\)

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\(^8\)Because candidate distributions are symmetric, an increase in heterogeneity also raises the probability that the party’s candidate will be positioned even farther away from the voter than the party platform. For this reason, increasing heterogeneity can never propel a party with a probability of winning that is less than 0.5 to a status where it is more likely than not to win. Since parties benefit from heterogeneity under these conditions, we would also expect them to benefit from selecting candidates whose positions diverge from the party platform when they have more control over the candidate distribution and can “target” candidates to specific districts.
Figure 3: The probability that party $i$ wins the election, for varying values of $\mu_i$ and $\sigma_i$. Both parties have normal distributions, and $\mu_j = 4$ and $\sigma_j = 2$. For each $\mu_i$, there is a unique value of $\sigma_i$ that maximizes party $i$’s probability of winning.\footnote{I use the \texttt{integrate} function in R to calculate the integral numerically. This adaptive algorithm has been shown to outperform other quadrature functions in R and S-PLUS (Kuonen 2003). I also verified a number of the computations using Mathematica.}
In one case it behooves a party to be perfectly homogeneous.

**Proposition 4.** If a party’s platform is at the position of the median voter, increasing heterogeneity will decrease its probability of winning.

(Formal Statement: If \( \mu_i = 0 \), then \( \frac{\partial P}{\partial \sigma_i} < 0 \) for all \( \sigma_i > 0 \).)

Proposition 4 is fairly intuitive. A party with a platform equal to the median voter maximizes its probability of winning when it is perfectly homogeneous. This is the case regardless of the other party’s distribution. The top line in Figure 3 depicts the probability of winning for a party with a platform at the median (i.e. \( \mu_i = 0 \)). As we can see, this line decreases for all values of \( \sigma_i \); as the party relaxes candidate entry and allows for greater heterogeneity, it suffers electorally. Taken with Proposition 1, this implies that if parties can freely move their platforms and adjust their heterogeneity, two homogeneous parties will converge to the position of the median voter. In equilibrium, the parties perfectly resemble those in the Downsian model.\(^{10}\)

**Proposition 5.** Holding the other party’s distribution constant, party \( i \)’s optimal level of heterogeneity increases with its platform’s distance from the median voter.

(Formal Statement: Suppose \( \sigma_i^* \) is the optimal level of heterogeneity for party \( i \). Then \( \sigma_i^* \) is increasing in \( |\mu_i| \).

Parties with off-median platforms can improve their odds of winning by selecting a heterogeneous candidate pool. All else equal, a party should become more heterogeneous as its platform moves away from the median voter’s ideal point.\(^{11}\) (In fact, every level of heterogeneity is optimal for some platform location: for any \( f_j \) and \( \hat{\sigma}_i \), there is a \( \mu_i \) for which \( \sigma_i^* = \hat{\sigma}_i \).) Thus, a party’s

\(^{10}\)Figure 3 also conveys the results from Propositions 1 and 2. For any fixed level of heterogeneity, party \( i \)’s probability of winning strictly decreases with its platform’s distance from the median voter (Proposition 1). Similarly, a shift in the position of the party platform has a greater effect when the platform is neither very close nor very far from the median voter (Proposition 2a). A change in platform has the smallest effect on election outcomes when heterogeneity is high (Proposition 2b).

\(^{11}\)Although it is increasing in distance, the optimal level of heterogeneity for the party closest to the median voter is very small (as shown in Figure 3).
heterogeneity strategy will depend on its popularity. If, for example, the congressional agenda forces a party’s members to take an unpopular stand, the leadership may respond by emphasizing the diversity of opinions within the party and recruiting a wide variety of candidates. If instead the party successfully moves toward the center of the voter distribution, it should hone its appeal and select candidates who adhere to the party line. Proposition 5 also suggests that the two parties should adopt different levels of heterogeneity when their platforms are not equidistant from the median voter.

**Proposition 6. The closer party should be more homogeneous than the farther party.**

When two parties are equidistant from the median voter, the more homogeneous party is more likely to win.

*(Formal Statement: If \( \mu_i \leq \mu_j \), then \( \frac{\partial P_i}{\partial \sigma_i} < 0 \) when \( \sigma_i \geq \sigma_j \).)*

When two parties are equidistant from the voter, they should both choose levels of heterogeneity that undercut their opponent. For example, if parties \( i \) and \( j \) have normal distributions where \( |\mu_i| = |\mu_j| = 4 \) and \( \sigma_j = 2 \), then party \( i \)’s best response is to choose \( \sigma_i = 1.55 \). Having observed \( \sigma_i = 1.55 \), however, party \( j \)’s best response is to decrease \( \sigma_j \) to 1.35. As parties observe their competitors’ distributions and adjust their own heterogeneity accordingly, this process will continue until both parties adopt zero heterogeneity. Thus, when two parties are equidistant from the voter (regardless of how far apart they are), they should both field a perfectly homogeneous set of candidates in equilibrium.

This equilibrium is very sensitive to small perturbations in platform locations. If there is any asymmetry in the parties’ positions, the farther party must increase its heterogeneity to have any chance of winning. For example, suppose that party \( i \)’s platform moves from 4 to 4.1, while party \( j \)’s platform remains at 4. Then, if both parties continue to be perfectly homogeneous, party \( j \) wins with certainty. Instead, party \( i \) should increase its heterogeneity; when \( \sigma_j = 0 \), party \( i \) should set
Proposition 6 implies that the party closer to the median voter should always opt for a less diverse candidate pool than the farther party. This is the case even when the more distant party adopts a suboptimal distribution, such as a very homogeneous candidate pool. The optimal heterogeneity for the farther party also depends on the standard deviation of the closer party, but it is not always smaller or greater than the closer party.

Proposition 7. For any pair of party platforms, there exists a pair of heterogeneity strategies that constitutes a Nash equilibrium. This equilibrium is unique except when exactly one party has a platform equal to the position of the median voter.

(Formal Statement: For any $\mu_i$ and $\mu_j$, there exists a $\sigma_i^*$ and $\sigma_j^*$ such that $\sigma_i^*$ is the optimal standard deviation for party $i$ when party $j$ has standard deviation $\sigma_j^*$ and vice versa; i.e. $(\sigma_i^*, \sigma_j^*)$ is a Nash equilibrium. This equilibrium is unique in all cases except when $\mu_i = 0$ and $\mu_j \neq 0$ or vice versa.)

Above we saw that when party platforms are equidistant from the median voter, the equilibrium strategy for both parties is perfect homogeneity. The proof of Proposition 7 demonstrates that we should also observe stability when parties are not equidistant. Combining Propositions 6 and 7 implies that in equilibrium the party closer to the median voter should recruit a more homogeneous pool of candidates than the party that is farther away.

This means that when political events alter the proximity of either party’s platform to the median voter, both parties should adjust their heterogeneity. As “partisan tides” shift the electorate in favor of one party or another, the party gaining in proximity to the median should capitalize on its platform and increase homogeneity. Conversely, the party that loses support should adopt the opposite strategy and diversify its candidate pool. After losing in 2008, the Republicans appear to have successfully pursued this strategy. Republican National Committee Chair Michael Steele

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12 When $\mu_i = 0$ and $\mu_j \neq 0$, $(0, \sigma_j^*)$ is a Nash equilibrium for any value of $\sigma_j^*$. 

refused to endorse candidates in House primary elections, and actively reached out to a variety of interests within the party, such as the Log Cabin gay-rights group and the Tea Party movement.

In other cases, party platforms change while the distribution of voters remains stable. Over the past 35 years, the Democratic and Republican Party platforms have increasingly moved toward opposite poles on the ideological spectrum (McCarty, Poole, and Rosenthal 2006, Theriault 2008). A number of factors may explain this recent polarization, including partisan redistricting (Carson et al. 2007), a rise in inequality (McCarty, Poole, and Rosenthal 2006), extreme party activists (Aldrich 2011, Theriault 2008), and increased voter sorting along ideological and partisan lines (Levendusky 2009).  

In this era of polarized partisanship, should parties adjust their heterogeneity? Proposition 8 suggests they should.

**Proposition 8. The more polarized the party platforms, the greater the party heterogeneity in equilibrium.**

(Formal Statement: Suppose \((\sigma_i^*, \sigma_j^*)\) is a Nash equilibrium for the pair of platforms \((\mu_i, \mu_j)\). Then \((k\sigma_i^*, k\sigma_j^*)\) is a Nash equilibrium for the platforms \((k\mu_i, k\mu_j)\) for any \(k \geq 0\).)

As the two parties’ platforms simultaneously move away from the median voter, parties should broaden their appeal and nominate candidates with a wider range of positions. Extreme conservatives and liberals will be as likely to run as moderate candidates, and both primary and general elections should prove more volatile.

Proposition 8 appears to contradict evidence that parties have become more coherent as they have moved apart (McCarty, Poole, and Rosenthal 2006). But there are several important differences between the model’s prediction and the empirical record. First, by polarization Proposition 8 refers specifically to a simultaneous shift in party platforms away from the median voter. As the proof demonstrates, increasing polarization (as it is defined here) is formally equivalent to changing

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13 See Theriault (2008) for a comprehensive discussion of the sources of polarization.
the scale on which parties and voters’ policy positions are measured.\textsuperscript{14} The level of overlap (that is, the probability that a Democrat is to the left of a Republican) remains constant. Second, the heterogeneity model describes the preferences of all candidates – not just those who win the election. Winners may disproportionately represent one branch of the party, and thus a more homogeneous pool. In addition, factors not included in this model may play an important role in determining both polarization and homogeneity.

Moreover, increased homogeneity may be explained by a growing symmetry in party platforms around the median voter. Recall from Propositions 6 and 7 that two equidistant parties will adopt perfect homogeneity in equilibrium (regardless of their absolute distance). As the two parties’ platforms become more evenly spaced around the median voter (by either the farther party moving closer to the median voter, or the closer party moving farther away), their average equilibrium heterogeneity levels will decline and approach zero.\textsuperscript{15} If the Republicans and Democrats are becoming increasingly symmetric around the median voter’s position as their platforms move apart, then their average level of heterogeneity should decline.

Testing this hypothesis requires a measure of the distance between each party’s platform and the median voter over time. Standard measures of polarization focus on the distance between the two parties, not their relative distances to the median voter. But, as the two parties grow more equidistant, we would expect elections to become more competitive. Volatility should be high, and majorities in the legislature should be small. This is consistent with the House election record over the past 17 years. After decades of Democratic dominance, control of the House is now anyone’s game. Since 1995, the majority party has held less than 55 percent of the seats on average, and

\textsuperscript{14}Ashworth and Bueno de Mesquita (2008) also define polarization as a simultaneous shift in party platforms, and find that polarization increases heterogeneity as a result of a similar change in scale. In their model, voters care less about uncertainty relative to platform positions as parties move apart, making (costly) screening less consequential. In Snyder and Ting (2002), heterogeneous parties choose to separate from one another in order to increase candidate homogeneity. In both models, homogeneity increases a party’s appeal to voters who are risk averse.

\textsuperscript{15}Note that this does not contradict Proposition 5. A party’s optimal heterogeneity is still increasing as its platform moves away from the median voter when its competitor’s distribution is held constant.
small shifts in voter preferences have produced large effects on election outcomes. I return to this in
the empirical section, but future research may investigate the relationship between party symmetry
around the median voter and heterogeneity.

Extensions

The heterogeneity model presented above depicts parties as collections of candidates with varying
positions rather than single actors with unique policy stances. I find that parties should strategically
diversify their candidate pool to compensate for off-median positions, and that there exists a Nash
equilibrium level of heterogeneity for any two party platform pairs such that the party closest to
the median is more homogeneous than its competitor. These findings may appear to depend on two
assumptions that could limit the model’s real world applicability. These are that voters care only
about the positions of the two candidates running in their district, and that voters observe these
positions with certainty. Thus, it is important to ask if the findings persist when the assumptions
are relaxed.

This section extends the model in three ways. I begin by allowing voters to observe the positions
of multiple candidates within each party, and then vote for the party with the most appealing set
of candidates. Second, I consider a case in which voters care about the positions of candidates
and party platforms. Voters weigh the benefit of selecting a member who well-represents their
position with one who will contribute to the legislative success of their preferred party. In both
of these extensions, I find that parties should increase their optimal heterogeneity relative to the
baseline model. The third extension incorporates voter uncertainty. Voters observe a signal about
candidate preferences – rather than a clear and fully informative description of future behavior –
making them reluctant to support a candidate whose position is vague or variable. Depending on
the relationship between uncertainty and heterogeneity, the closer or the farther party may benefit
from uncertainty, and optimal party heterogeneity may decrease or stay the same.

Each of these extensions alters a party’s optimal level of heterogeneity. Yet the primary findings persist: a unique level of heterogeneity maximizes a party’s success; this optimal level of heterogeneity is typically non-zero; and the closer party should be more homogeneous than the farther party.

**Multiple Candidates and Multiple Signals**

Up to now, voters have simply observed the positions of the two candidates running in their district. In reality, however, voters may learn about the positions of many different candidates in each party. For example, even in districts where a Republican candidate was not endorsed by the Tea Party, voters surely learned about the Tea Party’s position and presence in the Republican Party in 2010. Knowing that legislators must work together once they are in Congress, a voter may decide to vote for the party whose set of candidates is most sympathetic to their preferred policy instead of simply the candidate whose position is closest to their own (Austen-Smith 1984).

To incorporate this into the heterogeneity model, suppose that voters observe $N_p$ candidates’ positions for party $p$: \{c_{p1}, c_{p2}, \ldots, c_{pN_p}\}, where $N_p \geq 1$ and each candidate is independently drawn from the distribution, $f_{p}$. Voters select the candidate whose party has an average candidate position, $ar{c}_p = \frac{1}{N_p} \sum_{l=1}^{N_p} c_{pl}$, closest to their ideal policy. Because the standard deviation of the distribution of $\bar{c}_p$ is smaller than the standard deviation of the distribution of $c_p$, parties should increase their optimal heterogeneity when voters observe multiple candidates’ positions. For example, if candidate preferences in both parties follow normal distributions, parties should increase heterogeneity by a factor of $\sqrt{N_p}$. And if $(\sigma_i^*, \sigma_j^*)$ is a Nash equilibrium for the pair of platforms $(\mu_i, \mu_j)$ when voters observe a single candidate, then $(\sigma_i^* \sqrt{N_i}, \sigma_j^* \sqrt{N_j})$ is a Nash equilibrium when voters observe $N_i$ and $N_j$ candidates from parties $i$ and $j$, respectively. Optimal heterogeneity increases (at a decreasing
rate) with the number of candidates a voter observes.

Parties may also signal their platform location through campaign messages, television advertisements, or the behavior of elected officials. Voters may learn about the party’s position indirectly from activists or interest groups, or they may observe “accidental data” about the party’s position by overhearing a political conversation among strangers or driving along a highway dotted with political billboards (Levendusky 2009). If voters ultimately care only about a party’s platform, and a party’s signals around its platform follow a symmetric, single-peaked distribution, then the comparative statics in this and the previous section apply to party messages as well as candidate positions. When voters learn more about parties – due to high-profile campaigns, greater attention in the media, or heightened voter interest in politics – parties should broadcast a greater variety of messages about their policy position. Parties with platforms in line with the median voter should send more consistent signals about their positions, while those with distant platforms may find it advantageous to be ambiguous and emphasize the diversity of opinions within the party.  

Voters Observe Platforms and Candidates

Of course, voters most likely care about both a party’s platform (or the average position of its candidates) and the specific position of the candidate in their district. They will weigh the importance of electing a party that is likely to pursue legislation close to their ideal point and electing a candidate with whom they identify. Voters may also weigh party and candidate positions if they expect candidates to move toward their party’s median member once they are elected. To add party platforms to the model, suppose that a voter’s utility is based on their estimate of the party’s platform and the candidate that party nominates in their district:

16Previous research argues that candidates may be strategically ambiguous about their positions on unpopular issues by avoiding those issues altogether (Page 1976, Campbell 1983). Experimental evidence suggests that voters may prefer candidates who send ambiguous messages to those with precise policy positions (Tomz and Van Houweling 2009).
where $1 - \alpha$ is the weight the voter assigns to their estimate of the party’s platform, and $\alpha$ is the weight the voter assigns to the candidate’s position. When $\alpha = 1$, the model reduces to its original state, and a party’s optimal level of heterogeneity remains equal to $\sigma_p^*$. When $\alpha$ is less than one, the voter’s utility partially depends on their estimate of the party’s platform. As in the previous section, a party should increase its optimal level of heterogeneity in response. Parties set their highest level of heterogeneity when $\alpha = 0$, and the model is the same as in the previous section.\textsuperscript{17}

At first glance, this result may appear counterintuitive. When voters care more about the location of a party’s platform, parties should be more – not less – heterogeneous. Yet this is because the value of strategic heterogeneity diminishes as voters learn more about the party’s position. To retain heterogeneity as a tool, parties must increase the scope of their candidate pool.

**Voter Uncertainty**

Thus far, I have assumed that candidates’ positions (and in the previous sections, party signals) are accurate and credible. Yet in reality, candidates may obscure their position (strategically or not) by sending imprecise signals about their policy stances. Once in office, candidates may deviate from the positions they declared during their campaigns. Savvy voters will factor this uncertainty into their assessments of candidates and possibly alter their vote.

Previous research has argued that party heterogeneity is an electoral liability precisely because it increases uncertainty about candidate locations (Ashworth and Bueno de Mesquita 2008, Grynaviski 2010, Snyder and Ting 2002). If voters are risk averse, they may be more willing to vote for a distant candidate whose position is known than a candidate whose expected position is closer, but

\textsuperscript{17}Specifically, if candidate positions in party $p$ follow a normal distribution, party $p$ will optimize its heterogeneity when $\sigma_p = \sigma_p^*/\sqrt{\frac{1-\alpha^2}{N_p} + (\alpha)^2}$. 

23
Thus, this section asks: when candidate positions are uncertain and voters are risk averse, do parties revert to acting as homogeneous teams?

Suppose that a voter interprets the candidate’s position for party \( p \) as a probability distribution, \( g_p(x) \), where \( E[g_p(x)] = c_p \). The voter’s uncertainty about the candidate’s position is captured by the standard deviation, \( s_p \), of \( g_p(x) \). For simplicity, I model voter utility using a standard quadratic loss function:

\[
u(|v - c_p|) = -|v - c_p|^2.
\]

Thus, the median voter’s (\( v = 0 \)) expected utility from candidate \( c_p \) is

\[
EU(c_p) = - \int_{-\infty}^{\infty} g_p(x)x^2 \, dx = -c_p^2 - s_p^2,
\]

and they will vote for party \( i \) when

\[-c_i^2 - s_i^2 > -c_j^2 - s_j^2.\]

If both parties’ candidates have equal uncertainty (i.e. \( s_i = s_j \)), the results are identical to those of the model without uncertainty: voters simply support the candidate whose position most closely matches their own, and party strategies remain the same as in the baseline model.

If the uncertainty levels for the two parties are unequal, the party with greater uncertainty is hurt most. As uncertainty increases, the share of a voter’s utility that is based on distance decreases. Thus, if both parties face an equal probability of being more uncertain than one another, uncertainty will on average help the farther party more than the closer party.\(^{19}\)

\(^{18}\)In contrast with the experimental results mentioned previously, observational studies have typically found that voters behave in accordance with risk-averse utility functions (Alvarez 1998, Bartels 1986).

\(^{19}\)Specifically, suppose that \( s_p \sim \Theta \) for \( p \in \{i, j\} \), where \( \Theta \) is a distribution with non-negative support. Then, as \( \Theta \) increases (in the sense of first-order stochastic dominance), the farther party’s probability of winning improves and
Of course, a party’s uncertainty may be positively related to its heterogeneity, causing risk-averse voters to punish parties with wide-ranging candidate pools (Ashworth and Bueno de Mesquita 2008, Grynaviski 2010, Snyder and Ting 2002, Woon and Pope 2008). To incorporate this into the model, suppose that \( s(\sigma_p) \) is the uncertainty around \( c_p \), where \( s \) is an increasing function.\(^{20}\) In this case, the voter supports party \( i \) when

\[-c_i^2 - s(\sigma_i)^2 > -c_j^2 - s(\sigma_j)^2.\]

Intraparty heterogeneity may now be a partial liability. Although heterogeneity may increase a party’s chance of fielding a candidate close to the median voter, it may also hurt the party’s chance of winning when voters are risk averse. As important, the party with greater heterogeneity is hurt the most. The farther party – which is more heterogeneous in equilibrium when there is no uncertainty – faces greater incentives to reduce its optimal level of heterogeneity. The degree to which it will do so depends on the difference between \( s(\sigma_i) \) and \( s(\sigma_j) \).

For example, let \( s \) be a simple linear function: \( s(\sigma_p) = k\sigma_p \). Here, the constant \( k \) effectively captures a party’s cost of heterogeneity due to voter uncertainty.\(^ {21}\) For all \( k > 0 \), both parties should adopt a lower level of heterogeneity in comparison with the case when there is no voter uncertainty, and they should decrease heterogeneity more as \( k \) increases. Figure 4 illustrates this point by plotting the probability that party \( i \) wins the election for varying values of \( \sigma_i \) and \( k \).

\(^{20}\)Alternatively, heterogeneity could be negatively associated with uncertainty if voters believe that a candidate’s stated position is more credible when that candidate is nominated by a heterogeneous party. I do not explore this in the current paper.

\(^{21}\)Note that Snyder and Ting (2002) essentially assume that \( s(\sigma_p) = \sigma_p \).
uncertainty. The dashed or dotted lines depict the probability that party $i$ wins when $k$ takes on three positive values. As we can see, a positive value of $k$ increases the probability that the more homogeneous party wins the election. (To the left of $\sigma_i = 2$, party $i$ is more homogeneous than party $j$, and the probability that it wins the election increases in $k$. To the right of $\sigma_i = 2$, party $i$ is more heterogeneous than party $j$, and its probability of winning decreases in $k$.) Overall, the optimal level of heterogeneity is decreasing in $k$. This holds for both parties, and the comparative statics remain the same. The closer party should be more homogeneous than the farther party for any increasing function $s$.$^{22}$

[Figure 4 about here]

Thus, while extending the model in varying ways – adding multiple candidates, voters who care about party platforms, and uncertainty – changes a party’s optimal heterogeneity in important ways, the model’s primary predictions remain the same. In equilibrium, parties that fail to converge to the median voter should be somewhat heterogeneous, and the farther party should be more heterogeneous than the closer party. In the next section I test this prediction by examining the Democrats’ and Republicans’ platforms and heterogeneity in national and state elections.

$^{22}$The comparative statics also hold when $s$ is a function of multiple signals drawn from the party’s distribution, as in the first extension.
Figure 4: The probability party $i$ wins the election for varying values of $k$. Candidates in parties $i$ and $j$ are distributed normally with means five and three, respectively. The standard deviation for party $j$ is fixed at two.
Party Heterogeneity in State Legislative Elections

Do political parties behave according to the logic of the heterogeneity model developed above? We know that candidates’ positions vary within both of the major U.S. parties. But is this variation simply reproducing heterogeneity in voter preferences, or is it also indicative of a strategic, electoral calculus by party leaders?

The extent to which American parties are involved in selecting eligible candidates is itself an empirical question for which current research does not provide a definitive answer. While conventional wisdom holds that the U.S. primary system minimizes the role of parties in candidate selection, recent research suggests otherwise (Cohen et al. 2008, Frendreis, Gibson and Vertz 1990, Sanbonmatsu 2006). Party leaders work behind the scenes to shape the candidate pool. In a nationwide study, Sanbonmatsu (2006) finds that over 75 percent of state party and legislative caucus leaders are actively involved in recruiting candidates, providing endorsements in primaries, campaigning for incumbents’ renomination, or even dissuading candidates who are out-of-step with the party platform. As Sanbonmatsu argues, “the party may actively recruit precisely because its hands are tied at the primary stage” (2006, 45). The study also finds that party leaders from competitive states are significantly more active in recruitment, suggesting that they (at least think they can) shape the nomination process. To the extent these leaders are successful at influencing their parties’ candidate pools, we would expect the heterogeneity model’s predictions to be upheld in U.S. elections.23

This section tests the model’s prediction that parties closer to the median voter are more homogeneous than their more distant competitors. Ideally, a measure of heterogeneity would capture

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23Sanbonmatsu also finds that leaders are more active in recruitment, and more likely to “think outside the box,” when their parties are in the minority. “The minority party typically has fewer informal qualifications expected of its candidates... identifying nontraditional sources of candidates is much more important for the minority party” (Sanbonmatsu 2006, 57). This is largely because recruitment is more difficult for minority parties. Potential politicians may be uninterested in joining a party with little influence over state politics. But increased activity among minority party leaders may also reflect an electoral strategy to recruit a diverse candidate pool.
variation in preferences among a party’s potential candidates in a given district. Because this is virtually impossible at the sub-presidential level, I employ the next best option: measuring the heterogeneity across districts of those candidates who actually receive their party’s nomination. This requires the additional assumption that party leaders recruit and endorse the same types of candidates (i.e. candidates with the same distribution of preferences) across multiple districts, and that they choose their optimal level of heterogeneity based on the party’s average distance to the median voter across districts.\textsuperscript{24}

Accurately measuring the heterogeneity of candidates across districts also poses a challenge because it requires information about the positions of all candidates – winners and losers, incumbents and challengers. While winners establish reputations in office through roll-call voting and bill sponsorship, the larger group of unsuccessful candidates has no comparable policy record. Fortunately, Bonica (2010) has recently developed a method to rate the policy positions of all candidates with political action committee donation data. Using an item response theory count model, Bonica estimates ideal points, or \textit{cfscores}, for candidates at the federal and state level from 1990 to 2010.\textsuperscript{25} A visual inspection of the data confirms that the distribution of candidates in congressional and state parties is well-approximated by a single-peaked, symmetric distribution.

I assume that a party is closer to the median voter if it wins a majority of votes in a given congressional or state election. (Under equilibrium conditions, the party that is closer to the median voter will win more often than not.) A party’s heterogeneity is measured as the standard deviation

\textsuperscript{24}In reality, party leaders may focus their recruitment efforts on candidates whose preferences match those of the voters in each district. As long as a party’s ability to match candidates with districts is independent of its proximity to the median voter, the comparative statics of the heterogeneity model should still hold, and the closer party should be more homogeneous than the farther party.

\textsuperscript{25}Previous research has measured the positions of non-incumbents with candidate surveys (for example, see Ansolabehere, Snyder, and Stewart 2001 or Burden 2004a). But because these surveys only exist for a handful of election cycles, they cannot be used to systematically analyze changes in party heterogeneity. Burden (2004b) develops an alternative method for calculating challengers’ positions using election returns and Poole and Rosenthal’s NOMINATE scores (1997, 2007), but these data are not available for as many state elections. For more information on \textit{cfscores}, see Bonica’s “Ideology and Interests in the Political Marketplace” at files.nyu.edu/ajb454/public.
of its candidates’ cfscores. Because proximity may change as voter preferences or party positions shift between elections, I match a party’s heterogeneity in a given election with its majority or minority status resulting from (i.e. in the term following) the same election. To test the model’s predictions, I compare the number of cases in which the majority party is less heterogeneous than the minority party (i.e. those that support the hypothesis) with the number of cases in which the majority party is more heterogeneous than the minority party (i.e. those that do not support the hypothesis).26

In total there are 11 congressional elections and 350 state legislative elections in the data. The Democrats were the minority party in Congress for all but two terms (following the 1990 and 2008 elections), and they were more heterogeneous than the Republicans in all but one (following the 2000 election). Thus, eight of the 11 congressional observations fit with the expectation that the party farther from the median voter is more heterogeneous. Because there are only a handful of cases and almost no variation in majority status by party, I concentrate the following analysis on state elections.

Of the 350 state legislative elections, the minority party’s standard deviation is larger than that of the majority party in 218 cases, or 62 percent of the time. Table 1 presents the data for Democrats, disaggregated by majority or minority status and relative heterogeneity. The 218 cases in which the majority party has a lower standard deviation than the minority party include those instances in which the Democrats are in the majority and have a lower standard deviation (95) and when they are in the minority and have a higher standard deviation (123).

As we can see, the data strongly support the predictions of the model; the losing party –

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26I focus on the Nash equilibrium predictions of the model because they do not require a measure of the median voter’s location. While a party’s relative distance to the median voter is not monotonic in vote share, its ordinal distance (e.g. closer or farther) is monotonic. (Recall that a party can gain or lose votes as it moves closer to the median voter, depending on the other party’s behavior).
presumably farther from the median voter – is 1.6 times as likely to field a more diverse set of candidates than the winning party. This difference is statistically significant (\( p < 0.01 \), one-tailed test). Controlling for party strengthens the relationship between a party’s majority-minority status and its heterogeneity. A party’s probability of fielding more heterogeneous candidates than its competitor increases by .27 as it moves from majority to minority status (i.e. as it moves farther from the location of the median voter).

Table 1 also reveals that the Democrats were considerably more heterogeneous than the Republicans in most states throughout the time period, even though they were the majority party more than half the time. The Democrats fielded a wider variety of candidates than the Republicans 51 percent of the time when they were in the majority and 78 percent of the time when they were in the minority. This fits with our understanding of the two parties. The Democrats have traditionally included a broad coalition of members from varying economic and ideological positions. And while the Republican Party has historically been governed by a rigid organization and a formal set of rules, Democratic nominations more often resemble a “wide-open free-for-all” (Galvin 2010, 28).
Democrats: \( P(\text{Higher SD} \mid \text{Majority}) = \frac{97}{192} = .51 \)
\( P(\text{Higher SD} \mid \text{Minority}) = \frac{123}{158} = .78 \)

Republicans: \( P(\text{Higher SD} \mid \text{Majority}) = \frac{35}{158} = .22 \)
\( P(\text{Higher SD} \mid \text{Minority}) = \frac{95}{192} = .49 \)

Table 1: The number of state legislative elections in which Democrats were in the majority or minority and had a lower or higher standard deviation than the Republicans, and the probability that Democrats had a higher standard deviation than Republicans (and vice versa) given their majority or minority status. Majority status is a measure of proximity to the median voter.
This example provides an empirical test that strongly supports the model’s predictions. But it is important to note that many features of the electoral environment may affect a party’s heterogeneity as well as its majority status. I control for party in this analysis, but other factors that may affect party heterogeneity – such as state election laws or legislative rules – are not captured in the model and warrant future examination.

Although the data are not reported here, it is also worth noting that there is no obvious trend at either the congressional or state level in candidate heterogeneity. In Congress, as well as in most states, the mean positions of the candidates in the Democratic and Republican parties are moving apart from one another. But, unlike evidence based on the roll call votes of members in Congress, these data reveal no corresponding trend in party unity.

Conclusion

Political science research typically assumes or implies that party leaders prefer homogeneity. When their members are in agreement, majority party leaders can pass partisan legislation more efficiently (Rohde 1991). Minority party leaders are also better positioned to block opposing legislation when their members act as a coherent team. In the electoral arena, homogeneous parties offer more reliable signals about their candidates’ preferences and their platform’s location (Ashworth and Bueno de Mesquita 2008, Gryniviski 2010, Snyder and Ting 2002, Woon and Pope 2008). For risk-averse voters, greater certainty about a party’s position can make all the difference in which party to support (Alvarez 1998, Bartels 1986, Enelow and Hinich 1981). While party leaders cannot nominate every candidate who runs under their label, they can encourage some candidates more than others by offering campaign assistance, labor, or endorsements, or even by dissuading would-be candidates from running in the party primary. To the extent party leaders can influence candidate selection, political science offers a clear prescription: enforce homogeneity.
The heterogeneity model presented in this article largely agrees that homogeneity is the optimal strategy – *for parties close to the median voter*. Parties that are well positioned to win a majority of votes should recruit candidates whose positions align with the party platform. But a homogeneous party committed to a platform distant from the median voter is doomed to defeat. For parties that are out of step, increasing candidate heterogeneity may be their only viable option. By opening up their recruitment process, parties draw in candidates who may appeal to a greater share of the electorate.

Thus, while previous research argues that heterogeneity decreases a party’s election prospects, the model in this article finds that parties may choose to be strategically heterogeneous precisely because their platforms are unpopular. This presents an interesting asymmetry not found in the literature: while the party close to the median voter should be fairly homogeneous, the farther party should seek to diversify its candidate pool. The article also demonstrates that important extensions to the model – such as adding informative party labels or voter uncertainty – do not change the comparative statics. But they do change the magnitude of a party’s optimal heterogeneity. When voters are hyper-vigilant, the media is widespread, or campaigns are long and drawn out, parties should send more varied signals about their positions. Similarly, when voters care about the location of the party platform as well as of their own candidate, parties should become more heterogeneous. Voter uncertainty about candidates’ positions may cause parties to decrease or maintain their optimal level of heterogeneity.

Future research may extend the model in several ways. Because a party’s platform location and heterogeneity level are not independent, it is important to investigate the tradeoffs for strategic parties in altering either their platform or their heterogeneity. The political or institutional environment may make it easier for some parties to change their platform or heterogeneity than others. Majority parties can credibly claim to maintain the position they held in the previous term, while minority parties may enjoy more flexibility (but less credibility) to change their platform. Because
they have fewer incumbents, the minority party may also have more control over candidate heterogeneity. In addition, future research may add a candidate affiliation stage to the model, as in Ashworth and Bueno de Mesquita (2008) or Snyder and Ting (2002), or endogenize party platform locations. Similarly, the degree to which party heterogeneity reflects a party’s proximity to the median voter should depend on the power that party and legislative caucus leaders have over recruitment. It would be fruitful to examine how state laws and party organizational rules shape the role of party leaders in influencing platform location and candidate heterogeneity, and thus electoral success.

Appendix

Proof of Proposition 1

Equation (2) can be rewritten as $\int_{-\infty}^{\infty} f_j(c_j) \left[ \Phi\left( \frac{|c_j| - \mu_i}{\sigma_i} \right) - \Phi\left( \frac{-|c_j| - \mu_i}{\sigma_i} \right) \right] dc_j$, where $\Phi$ is the antiderivative of $\phi$. Taking the derivative with respect to $\mu_i$ gives

$$\frac{\partial P_j}{\partial \mu_i} = \int_{-\infty}^{\infty} f_j(c_j) \left( -\frac{1}{\sigma_i} \right) \left[ \phi\left( \frac{|c_j| - \mu_i}{\sigma_i} \right) - \phi\left( \frac{-|c_j| - \mu_i}{\sigma_i} \right) \right] dc_j. \quad (3)$$

Because $\phi(x)$ has mean zero, when $\mu_i > 0$, $\phi\left( \frac{|c_j| - \mu_i}{\sigma_i} \right) > \phi\left( \frac{-|c_j| - \mu_i}{\sigma_i} \right)$, and equation (3) is negative. Similarly, when $\mu_i < 0$, $\phi\left( \frac{|c_j| - \mu_i}{\sigma_i} \right) < \phi\left( \frac{-|c_j| - \mu_i}{\sigma_i} \right)$, and equation (3) is positive. Thus, the marginal effect of an increase in $|\mu_i|$ is negative, and the marginal effect of a decrease in $|\mu_i|$ is positive. Since $P_j = 1 - P_i$, the marginal effect of a decrease in $|\mu_i|$ is negative for party $j$. □
Proof of Proposition 2

Proposition 2a follows directly from substituting $|\mu_i| = 0$, or taking the limit as $|\mu_i|$ approaches infinity, in equation (3). Likewise, the first part of Proposition 2b follows directly from taking the limit as $\sigma_i$ approaches infinity in equation (3).

For the second part of Proposition 2b, suppose that $\sigma_i = 0$. Then $c_i = \mu_i$, and

$$P_i = 1 - \int_{-|\mu_i|}^{|\mu_i|} f_j(c_j) \, dc_j = 1 - \left[ \int_{-|\mu_i|}^0 f_j(c_j) \, dc_j + \int_0^{|\mu_i|} f_j(c_j) \, dc_j \right]$$

$$= 1 + \left[ \int_0^{-|\mu_i|} f_j(c_j) \, dc_j - \int_0^{|\mu_i|} f_j(c_j) \, dc_j \right].$$

(4)

Taking the derivative of equation (4) with respect to $|\mu_i|$ gives us:

$$\frac{\partial P_i}{\partial |\mu_i|} = -f_j(-|\mu_i|) - f_j(|\mu_i|) = -2f_j(|\mu_i|).$$

(5)

Because the derivative of $P_i$ with respect to $\mu_i$ is continuous as a function of $\sigma_i$, $\frac{\partial P_i}{\partial |\mu_i|}$ approaches $-2f_j(|\mu_i|)$ as $\sigma_i$ approaches zero.

Note that the result in Proposition 2a when $|\mu_i|$ approaches zero only holds when $\sigma_i > 0$. When $\sigma_i = 0$, the marginal effect of an increase in $|\mu_i|$ at $\mu_i = 0$ is given by substituting $\mu_i = 0$ into equation (5). □

Proof of Proposition 3

I begin by proving existence, and then I prove uniqueness of a global max. Since throughout the proof I am concerned with the effect of $\sigma_i$ on $P_i$, I treat $P_i$ as a single variable function of $\sigma_i$, $P_i(\sigma_i)$, and let $P'_i(\sigma_i)$ denote the derivative of $P_i$ with respect to $\sigma_i$.

If $\sigma_i = \sigma_j$, then party $i$ wins the election with some positive probability $p$. Since the $\lim_{\sigma_i \to \infty} P_i(\sigma_i) = 0$, there exists an $S$ such that if $\sigma_i > S$ then $P_i(\sigma_i) < p$. Because $P_i(\sigma_i)$ is continuous on the com-
pact interval $[0, S]$, it must have at least one global maximum on $[0, S]$. This maximum must be a global maximum since $P_i(\sigma_i) < p$ for $\sigma_i \notin [0, S]$, but $P_i(\sigma_i) = p$ for some $\sigma_i \in [0, S]$. 

Now I turn to the proof of uniqueness. Let $P_i(\sigma_i|c_j)$ denote the probability that party $i$ wins the election as a function of $\sigma_i$ given a fixed position, $c_j$, for the opposing candidate, 

$$P_i(\sigma_i|c_j) = \int_{-|c_j|}^{\sigma_i} \frac{1}{\sigma_i^2} \phi \left( \frac{c_j - \mu_i - \sigma_i^2 \phi}{\sigma_i} \right) \, dc_i = \Phi \left( \frac{|c_j| - \mu_i}{\sigma_i} \right) - \Phi \left( \frac{|c_j| - \mu_i}{\sigma_i} \right).$$

For a given $c_j$, the marginal effect of party $i$’s heterogeneity on the probability that party $i$ wins is 

$$P'_i(\sigma_i|c_j) = \frac{-|c_j| + \mu_i}{\sigma_i^2} \phi \left( \frac{|c_j| - \mu_i}{\sigma_i} \right) - \frac{|c_j| + \mu_i}{\sigma_i^2} \phi \left( \frac{|c_j| - \mu_i}{\sigma_i} \right).$$

(6) 

Because $\phi$ is symmetric around zero, this can be rewritten as 

$$P'_i(\sigma_i|c_j) = \frac{\mu_i - |c_j|}{\sigma_i^2} \phi \left( \frac{\mu_i - |c_j|}{\sigma_i} \right) - \frac{\mu_i + |c_j|}{\sigma_i^2} \phi \left( \frac{\mu_i + |c_j|}{\sigma_i} \right).$$

(7) 

Setting equation (7) equal to zero and multiplying both sides by $\sigma_i$, the first order condition is given by 

$$\frac{\mu_i - |c_j|}{\sigma_i} \phi \left( \frac{\mu_i - |c_j|}{\sigma_i} \right) - \frac{\mu_i + |c_j|}{\sigma_i} \phi \left( \frac{\mu_i + |c_j|}{\sigma_i} \right) = 0.$$ 

(8) 

Rearranging terms, equation (8) becomes 

$$\frac{(\mu_i + |c_j|)}{(\mu_i - |c_j|)} = \frac{\phi \left( \frac{\mu_i - |c_j|}{\sigma_i} \right)}{\phi \left( \frac{\mu_i + |c_j|}{\sigma_i} \right)}. $$

(9) 

By assumption the right hand side of equation (9) is strictly monotonic as a function of $\sigma_i$ and therefore there is at most one $\sigma_i$ satisfying the equation. If there is no such solution, then equation (6) must be decreasing for all $\sigma_i$, so $P_i(\sigma_i|c_j)$ is maximized at $\sigma_i = 0$. If there is a solution, it is
unique, and thus there is a unique value of \( \sigma_i \) that maximizes \( P_i(\sigma_i|c_j) \). Let \( \sigma^*_i|c_j \) denote this unique maximum. Then, because \( f_j \) is symmetric and single-peaked, and because \( \sigma^*_i|c_j \) is continuous and monotonic in \( c_j \), there exists a unique \( \sigma^*_i \) that maximizes \( P_i \). Numerical computations confirm this; for examples, see Figure 3.

Proof of Proposition 4

Proposition 4 follows directly from evaluating equation (7) at \( \mu_i = 0 \).

Proof of Proposition 5

Recall from the proof of Proposition 3 that \( \sigma^*_i|c_j \) is either zero or the unique solution to equation (9). If \( |c_j| \geq |\mu_i| \) then \( \sigma^*_i|c_j = 0 \) so the proposition holds trivially. Below I prove that \( \sigma^*_i|c_j \) is increasing in \( |\mu_i| \) for the case where \( |c_j| < |\mu_i| \) when \( \mu_i > 0 \). The case when \( \mu_i < 0 \) is similar.

Let \( |\mu_1| < |\mu_2| \), and let \( \sigma_i \) and \( \sigma_j \) be the optimal heterogeneity levels given \( \mu_i = \mu_1 \) and \( \mu_i = \mu_2 \), respectively. Then \( \sigma_1 \) satisfies

\[
\frac{\mu_1 + |c_j|}{\mu_1 - |c_j|} = \frac{\phi \left( \frac{\mu_1 - |c_j|}{\sigma_1} \right)}{\phi \left( \frac{\mu_1 + |c_j|}{\sigma_1} \right)},
\]

and \( \sigma_2 \) satisfies

\[
\frac{\mu_2 + |c_j|}{\mu_2 - |c_j|} = \frac{\phi \left( \frac{\mu_2 - |c_j|}{\sigma_2} \right)}{\phi \left( \frac{\mu_2 + |c_j|}{\sigma_2} \right)}.
\]

To show that \( \sigma_1 < \sigma_2 \), I introduce the intermediate point \( s \), defined by

\[
\frac{\mu_2 + |c_j|}{\mu_2 - |c_j|} = \frac{\phi \left( \frac{\mu_1 - |c_j|}{s} \right)}{\phi \left( \frac{\mu_1 + |c_j|}{s} \right)}.
\]
Since \( \frac{\mu + |c_j|}{\mu - |c_j|} \) is decreasing in \( \mu \) and \( \frac{\mu - |c_j|}{\mu + |c_j|} \) is decreasing in \( \sigma, s > \sigma_1 \). Then, since \( \frac{\mu - |c_j|}{\mu + |c_j|} \) is increasing in \( \mu, s < \sigma_2 \). Therefore \( \sigma_1 < \sigma_2 \). Numerical computations confirm that this extends to any symmetric, single-peaked distribution of \( c_j \).

\[ \int_{-\infty}^{\sigma_j} \phi(\mu-|c_j|) \, dc_j + \int_{\mu_i}^{\infty} \phi(\mu+|c_j|) \, dc_j = \int_{-\infty}^{\sigma_i} \phi(\mu-|c_j|) \, dc_j + 0.5 > 0.5 \] Because \( P_i \) is a single-peaked function of \( \sigma_i \), the value of \( \sigma_i \) that maximizes \( P_i \) must be strictly less than \( \sigma_j \). Since Proposition 5 demonstrates that \( \sigma_i^* \) is increasing in \( |\mu_i| \) (holding \( f_j \) constant), \( \sigma_i^* \) must also be less than \( \sigma_j \) when \( |\mu_i| < |\mu_j| \).

\[ \mu_i = \mu_j \text{ and } \sigma_j > 0 \]. Then, if \( \sigma_i = \sigma_j \), \( P_i = 0.5 \). If instead \( \sigma_i = 0 \), then \( P_i = \int_{-\infty}^{-|\mu_j|} f_j(c_j) \, dc_j + \int_{\mu_i}^{\infty} f_j(c_j) \, dc_j = \int_{-\infty}^{-|\mu_j|} f_j(c_j) \, dc_j + 0.5 > 0.5 \). Because \( P_i \) is a single-peaked function of \( \sigma_i \), the value of \( \sigma_i \) that maximizes \( P_i \) must be strictly less than \( \sigma_j \). Since Proposition 5 demonstrates that \( \sigma_i^* \) is increasing in \( |\mu_i| \) (holding \( f_j \) constant), \( \sigma_i^* \) must also be less than \( \sigma_j \) when \( |\mu_i| < |\mu_j| \).

\[ \mu_i = \mu_j \text{ and } \sigma_j > 0 \]. Then, if \( \sigma_i = \sigma_j \), \( P_i = 0.5 \). If instead \( \sigma_i = 0 \), then \( P_i = \int_{-\infty}^{-|\mu_j|} f_j(c_j) \, dc_j + \int_{\mu_i}^{\infty} f_j(c_j) \, dc_j = \int_{-\infty}^{-|\mu_j|} f_j(c_j) \, dc_j + 0.5 > 0.5 \). Because \( P_i \) is a single-peaked function of \( \sigma_i \), the value of \( \sigma_i \) that maximizes \( P_i \) must be strictly less than \( \sigma_j \). Since Proposition 5 demonstrates that \( \sigma_i^* \) is increasing in \( |\mu_i| \) (holding \( f_j \) constant), \( \sigma_i^* \) must also be less than \( \sigma_j \) when \( |\mu_i| < |\mu_j| \).

Proof of Proposition 6

Suppose \( \mu_i = \mu_j \) and \( \sigma_j > 0 \). Then, if \( \sigma_i = \sigma_j \), \( P_i = 0.5 \). If instead \( \sigma_i = 0 \), then \( P_i = \int_{-\infty}^{-|\mu_j|} f_j(c_j) \, dc_j + \int_{\mu_i}^{\infty} f_j(c_j) \, dc_j = \int_{-\infty}^{-|\mu_j|} f_j(c_j) \, dc_j + 0.5 > 0.5 \). Because \( P_i \) is a single-peaked function of \( \sigma_i \), the value of \( \sigma_i \) that maximizes \( P_i \) must be strictly less than \( \sigma_j \). Since Proposition 5 demonstrates that \( \sigma_i^* \) is increasing in \( |\mu_i| \) (holding \( f_j \) constant), \( \sigma_i^* \) must also be less than \( \sigma_j \) when \( |\mu_i| < |\mu_j| \).

Proof of Proposition 7

When \( \mu_i = \mu_j \), Proposition 6 implies that \( \sigma_i = \sigma_j = 0 \) is a Nash equilibrium. If \( \mu_i = 0 < |\mu_j| \) then \( (\sigma_i^*, \sigma_j^*) = (0, \sigma_j^*) \) is a Nash equilibrium for any value of \( \sigma_j^* \). Suppose \( 0 < |\mu_i| < |\mu_j| \). Let \( BR_i : [0, \infty) \to [0, \infty) \) be defined as party \( i \)'s best response given \( \sigma_j \): \( BR_i(\sigma_j) = \sigma_i^* \). Similarly, define \( BR_j : [0, \infty) \to [0, \infty) \) by \( BR_j(\sigma_i) = \sigma_j^* \). Define the curves \( I \) and \( J \) in \([0, \infty) \times [0, \infty) \) by \( I = \{(\sigma_i, \sigma_j) | \sigma_i = BR_i(\sigma_j) \} \) and \( J = \{(\sigma_i, \sigma_j) | \sigma_j = BR_j(\sigma_i) \} \). Then any intersection of \( I \) and \( J \) is a Nash equilibrium.

By Proposition 6, the curve \( I \) is contained in \( \{(\sigma_i, \sigma_j) | \sigma_i \leq \sigma_j \} \). If \( \sigma_i = 0 \) then the best response for party \( j \) is to adopt some non-zero heterogeneity, \( \sigma_j > 0 \) (otherwise party \( i \) wins the election with certainty). As \( \sigma_i \to \infty \), \( BR_j(\sigma_i) \to 0 \). Since the curve \( J \) is continuous and goes from the point \((0, \sigma_j)\) on one end towards the limit \((\infty, 0)\) at the other, there must exist a point \((s_i, s_j)\) on \( J \) such that \( s_i = s_j \). Let \( X \) denote the closed subset of \((\sigma_i, \sigma_j) \in [0, \infty) \times [0, \infty) \) bounded by \( \sigma_i = 0 \), \( \sigma_i = \sigma_j \), and the segment of \( J \) connecting \((0, \sigma_j)\) and \((s_i, s_j)\). Since \( BR_i(0) = 0 \) the
are Nash equilibria and \( \sigma \) pair, \((\sigma_1, \sigma_2)\) and \( \sigma_j \) sufficiently large the point \((BR_j(\sigma_j), \sigma_j)\) \( \in \mathcal{I} \) is outside of \( X \). Therefore, since \( \mathcal{I} \) is continuous, this implies that \( \mathcal{I} \) crosses the boundary of \( X \). However, \( \mathcal{I} \) cannot cross \( \sigma_i = 0 \) since every point of \( \mathcal{I} \) has \( \sigma_i \geq 0 \), and \( \mathcal{I} \) cannot cross \( \sigma_i = \sigma_j \), since every point of \( \mathcal{I} \) has \( \sigma_i < \sigma_j \). Therefore, \( \mathcal{I} \) must cross some part of the remaining boundary of \( X \), which consists of the points of \( J \) with \( \sigma_i \leq \sigma_j \). Thus, \( \mathcal{I} \cap J \neq \emptyset \), so a Nash equilibrium exists.

Next, I prove that this Nash equilibrium is unique except in the case where exactly one party has a platform equal to the position of the median voter. Because the case where \( \mu_i = \mu_j = 0 \) has already been shown to have a unique equilibrium at \((0, 0)\), I will assume from here on that neither party’s platform is positioned at the ideal point of the median voter. Suppose that for the platform pair, \((\mu_i, \mu_j)\), there are two Nash equilibria: \((\sigma_{i1}, \sigma_{j1})\) and \((\sigma_{i2}, \sigma_{j2})\). Proposition 3 demonstrates that \( \sigma^*_i \) is unique given \( \mu_i \) and \( f_j \), except when \( f_j = \delta(0) \). Then, if both \((\sigma_{i1}, \sigma_{j1})\) and \((\sigma_{i2}, \sigma_{j2})\) are Nash equilibria and \( \sigma_{i1} = \sigma_{j2} \), then \( f_j = \delta(0) \). Similarly, if \( \sigma_{j1} = \sigma_{j2} \), then \( f_i = \delta(0) \). Thus, if there are two Nash equilibria and neither party’s platform is positioned at the median voter, then it must be true that \( \sigma_{i1} \neq \sigma_{i2} \) and \( \sigma_{j1} \neq \sigma_{j2} \).

Given this, there are two possible cases: (1) \( P_i \) is greater at \((\sigma_{i1}, \sigma_{j1})\) than \((\sigma_{i2}, \sigma_{j2})\) or vice versa, or (2) \( P_i \) is equal at both \((\sigma_{i1}, \sigma_{j1})\) and \((\sigma_{i2}, \sigma_{j2})\). I will show that in both cases at least one party has an incentive to deviate.

For the first case, let us suppose without loss of generality that \( P_i(\sigma_{i1}, \sigma_{j1}) > P_i(\sigma_{i2}, \sigma_{j2}) \). Because \((\sigma_{i1}, \sigma_{j1})\) is a Nash equilibrium, it must be the case that \( P_j(\sigma_{i1}, \sigma_{j1}) \leq P_j(\sigma_{i1}, \sigma_{j'}) \) for all \( \sigma'_{j} \neq \sigma_{j1} \). This is equivalent to stating that \( P_i(\sigma_{i1}, \sigma_{j1}) \leq P_i(\sigma_{i1}, \sigma'_{j}) \). Because \( P_i(\sigma_{i1}, \sigma_{j1}) > P_i(\sigma_{i2}, \sigma_{j2}) \) and \( P_i(\sigma_{i1}, \sigma_{j1}) \leq P_i(\sigma_{i1}, \sigma'_{j}) \), it must be the case that \( P_i(\sigma_{i1}, \sigma'_{j}) > P_i(\sigma_{i2}, \sigma_{j2}) \), which implies that \( P_i(\sigma_{i1}, \sigma_{j2}) > P_i(\sigma_{i2}, \sigma_{j2}) \). Thus, party \( i \) has an incentive to deviate from \((\sigma_{i2}, \sigma_{j2})\), and it is not a Nash equilibrium.

Now suppose that \( P_i(\sigma_{i1}, \sigma_{j1}) = P_i(\sigma_{i2}, \sigma_{j2}) \). Then because \((\sigma_{i2}, \sigma_{j2})\) is an equilibrium, \( P_i(\sigma_{i2}, \sigma_{j2}) \geq P_i(\sigma'_{i}, \sigma_{j2}) \) for all \( \sigma'_{i} \neq \sigma_{i2} \). This implies that \( P_i(\sigma_{i2}, \sigma_{j2}) \geq P_i(\sigma_{i1}, \sigma_{j2}) \), which is equivalent to saying that
\[ P_j(\sigma_{i2}, \sigma_{j2}) \leq P_j(\sigma_{i1}, \sigma_{j2}). \]  
Because \( P_i(\sigma_{i1}, \sigma_{j1}) = P_i(\sigma_{i2}, \sigma_{j2}) \), it is also true that \( P_j(\sigma_{i1}, \sigma_{j1}) = P_j(\sigma_{i2}, \sigma_{j2}) \). Thus,

\[ P_j(\sigma_{i1}, \sigma_{j1}) \leq P_j(\sigma_{i1}, \sigma_{j2}). \]  

(10)

If the inequality in equation (10) is strict, party \( j \) has an incentive to deviate from \((\sigma_{i1}, \sigma_{j1})\), and it is not an equilibrium. If the two terms in equation (10) are equal, then \((\sigma_{i1}, \sigma_{j2})\) is also an equilibrium. But, as shown above, this is only possible if \( f_i = \delta(0) \), which contradicts the assumption that neither party is positioned at the ideal point of the median voter.  

\[ \square \]

**Proof of Proposition 8**

Let \( P_i^k(\sigma_i, \sigma_j) \) denote the probability that party \( i \) wins when parties \( i \) and \( j \) adopt platforms \( k \mu_i \) and \( k \mu_j \), and heterogeneity levels, \( \sigma_i \) and \( \sigma_j \), respectively. The proof proceeds by demonstrating that \( P_i(\sigma_i, \sigma_j) = P_i^k(k\sigma_i, k\sigma_j) \). Substituting \( k \mu_i \) and \( k \mu_j \) into the equation for \( P_i \) gives

\[
P_i^k(k\sigma_i, k\sigma_j) = \int_{-\infty}^{\infty} \frac{1}{k\sigma_j} \phi\left( \frac{c_j - k\mu_j}{k\sigma_j} \right) \int_{-|c_j|}^{|c_j|} \frac{1}{k\sigma_i} \phi\left( \frac{c_i - k\mu_i}{k\sigma_i} \right) \, dc_i \, dc_j.
\]

In the second integral, changing variables using the substitution \( u = c_i/k \) gives

\[
P_i^k(k\sigma_i, k\sigma_j) = \int_{-\infty}^{\infty} \frac{1}{k\sigma_j} \phi\left( \frac{c_j - k\mu_j}{k\sigma_j} \right) \int_{-|c_j|}^{|c_j|} \frac{1}{\sigma_i} \phi\left( \frac{u - \mu_i}{\sigma_i} \right) \, du \, dc_j.
\]

Now, changing variables in the first integral using the substitution \( w = c_j/k \) gives

\[
P_i^k(k\sigma_i, k\sigma_j) = \int_{-\infty}^{\infty} \frac{1}{\sigma_j} \phi\left( \frac{w - \mu_j}{\sigma_j} \right) \int_{-|w|}^{|w|} \frac{1}{\sigma_i} \phi\left( \frac{u - \mu_i}{\sigma_i} \right) \, du \, dw,
\]

which is equal to \( P_i(\sigma_i, \sigma_j) \).  

\[ \square \]
References


